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EFFECTIVE THERMAL CONDUCTIVITY COEFFICIENTS OF A GRAINY MEDIUM

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An exact solution is presented for the problem of determination of effective thermal conductivity coefficients of a composite medium with regularly spaced spherical grains.

1. The grainy layer considered consists of an isotropic matrix and spherical grains of radius R, the centers of which form a three-dimensional orthogonal lattice with periods a, b, and c. We denote by λ_1 and λ_2 the thermal conductivities of the matrix and grain materials. We introduce a Cartesian coordinate system x, y, z such that its origin coincides with the center O_{pqs} of an arbitrary pqs-th grain are pa, qb, sc (where p, q, s = 0, ±1, ±2, ...). We denote the temperature field within the matrix by T(x, y, z), and within the pqs-th grain by $T_{pqs}(r_{pqs}, \theta_{pqs}, \varphi_{pqs})$. Here $r_{pqs}, \theta_{pqs}, \varphi_{pqs}$ are spherical coordinates corresponding to the above Cartesian system and $x_{pqs} = x - pa$, $y_{pqs} = y - qb$, $z_{pqs} = z - sc$ ($x_{ooo} \equiv x$, $y_{ooo} \equiv y$, $z_{ooo} \equiv z$).

The problem consists of integrating the Laplace equation

$$\Delta t = 0 \tag{1}$$

at t = T in the volume outside the grains and $t = T_{pqs}$ in the region occupied by the pqs-th grain under the condition of ideal thermal contact between matrix and grains:

$$T = T_{pqs},\ \lambda_1 \frac{\partial}{\partial r_{pqs}} T = \lambda_2 \frac{\partial}{\partial r_{pqs}} T_{pqs};\ r_{pqs} = R. \tag{2}$$

We will first determine the effective thermal conductivity of the grainy layer in the z direction.

The essence of the method to be employed is the construction of an explicit expression for the temperature T(x, y, z) in a layer of thickness c, selected from the grainy medium. The faces of this layer are perpendicular to the z axis, and it contains a double-period system of grains with indices pq0. The upper face is removed from the plane xy by a distance h_1, and the lower, by a distance h_2, so that $h_1 + h_2 = c$ and $h_1, h_2 > R$.

We write the temperature field within the layer in the form

$$T = \gamma z + T_1(x, y, z), \tag{3}$$

where T_1 is a periodic function of arguments x and y with periods a and b, respectively,

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and γ is some constant. We require that solution (3) satisfy the following conditions:

$$T(x, y, h_1) - T(x, y, -h_2) = d, \frac{\partial}{\partial z} T(x, y, z)|_{z=h_1} = \frac{\partial}{\partial z} T(x, y, z)|_{z=-h_2}. \quad (4)$$

Here d is a constant. The temperature within the grains is expressed by a series

$$T_{pq} = \sum_{\nu=0}^{\infty} \sum_{\mu=-\nu}^{\nu} D_{\nu\mu}^{(pq)} r_{pq}^{\nu} X_{\nu}^{\mu}(\theta_{pq}, \varphi_{pq}), \quad (5)$$

where

$$X_{\nu}^{\mu} = p_{\nu}^{\mu}(\cos\theta_{pq}) \exp(i\mu\varphi_{pq}); \; p_{\nu}^{\mu}(u) = \sqrt{\frac{(2\nu+1)(\nu-\mu)!}{2(\nu+\mu)!}} \frac{(1-u^2)^{\frac{\mu}{2}}}{2^{\nu}\nu!} \frac{d^{\nu+\mu}}{du^{\nu+\mu}} (u^2-1)^{\nu}$$

are orthonormal Legendre joining functions; $D_{\nu\mu}^{(pq)}$ are undetermined constants. In Eq. (5) the third subscript s = 0 has been omitted to simplify the notation.

To obtain the double-periodic solution T_1 of Eq. (1) we construct a corresponding system of external solutions of the Laplace equation, decreasing as $|z| \to \infty$.

2. We will consider the harmonic function

$$t_0 = \sum_{p,q} \frac{1}{r_{pq}^2} p_1(\cos\theta_{pq}), \quad (6)$$

where $p_1(\cos\theta_{pq}) = \sqrt{\frac{3}{2}} \frac{z}{r_{pq}}$ is an orthonormal Legendre polynomial of first order, and the summation indices p and q vary from $-\infty$ to ∞. The function t_0 is represented by a double absolutely convergent series, satisfies the condition of double periodicity in x and y required, and is an odd function with respect to z. Moreover, $t_0 \to 0$ as $|z| \to \infty$. Therefore, there must exist for Eq. (6) another representation:

$$t_0 = \begin{cases} \sum_{m,n} a_{mn} \exp[-\delta_{mn} z + i(\alpha_m x + \beta_n y)], & z > 0, \\ -\sum_{m,n} a_{mn} \exp[\delta_{mn} z + i(\alpha_m x + \beta_n y)], & z < 0, \end{cases} \quad (7)$$

where $\alpha_m = 2\pi m/a$; $\beta_n = 2\pi n/b$; $\delta_{mn} = \sqrt{\alpha_m^2 + \beta_n^2}$.

By comparing Eqs. (6) and (7), we obtain an equation for determination of the unknown coefficients:

$$ab a_{mn} \exp(-\delta_{mn} z) = \sqrt{\frac{3}{2}}\, z \sum_{p,q} \int_0^a \int_0^b \frac{1}{r_{pq}^3} \exp[-i(\alpha_m x + \beta_n y)]\, dx dy,$$

which can easily be transformed to the form

$$ab a_{mn} \exp(-\delta_{mn} z) = \sqrt{\frac{3}{2}}\, z \iint_{-\infty}^{\infty} \frac{\exp[-i(\alpha_m u + \beta_n v)]}{(u^2 + v^2 + z^2)^{3/2}}\, du dv \quad (z > 0).$$

To calculate the integrals on the right side of this equation we introduce new integration variables using the formulas

$$u = \rho\cos\varphi, \; v = \rho\sin\varphi, \; du dv \sim \rho d\rho d\varphi.$$

Finally we have

$$ab a_{mn} \exp(-\delta_{mn} z) = \sqrt{\frac{3}{2}}\, z \int_0^{2\pi} d\varphi \int_0^{\infty} \frac{\exp[-i\rho\delta_{mn}\cos(\varphi - \varepsilon_{mn})]}{(\rho^2 + z^2)^{3/2}} \rho d\rho,$$

$$\exp(i\varepsilon_{mn}) = \frac{1}{\delta_{mn}} (\alpha_m + i\beta_n). \quad (8)$$

Integration in Eq. (8) is simple to perform for the case m = n = 0; the result has the form

$ab\alpha_{00} = \sqrt{6}\pi$. For the remaining values of the indices m and n we expand the integrand in a series of Bessel functions:

$$\exp[-i\rho\delta_{mn}\cos(\varphi-\varepsilon_{mn})] = \sum_k (-i)^k J_k(\delta_{mn}\rho)\exp[ik(\varphi-\varepsilon_{mn})].$$

This expression permits transformation of Eq. (8) to the form

$$ab\alpha_{mn}\exp(-\delta_{mn}z) = \sqrt{6}\pi z\int_0^\infty \frac{J_0(\rho\delta_{mn})}{(\rho^2+z^2)^{k03}}\rho d\rho. \tag{9}$$

The integral on the right side of Eq. (9) is the transform of a zero order Hankel function $(1/z)\exp(-\delta_{mn}z)$ [1]. Considering this fact, we obtain the following expression for the coefficients α_{mn} in Eq. (7):

$$a_{mn} = \frac{\sqrt{6}\pi}{ab}(m,\ n=0,\ \pm 1,\ \pm 2,\ldots). \tag{10}$$

Commencing from Eqs. (6), (7), (10) we construct some system of solutions of Eq. (1) having the same properties of periodicity and decay as $|z| \to \infty$ as does t_0. For this purpose we make use of the following relationships of harmonic function theory:

$$\begin{gathered}\frac{1}{r_{pq}^{\nu+1}}X_\nu^\mu(\theta,\ \varphi) = D_{\nu\mu}\left[\frac{1}{r^2}\ p_1(\cos\theta)\right],\\ D_{\nu\mu} = (-1)^{\nu-1}\sqrt{\frac{2\nu+1}{3(\nu+\mu)!\ (\nu-\mu)!}}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^\mu\left(\frac{\partial}{\partial z}\right)^{\nu-\mu-1}\\ (\nu=1,\ 2,\ \ldots;\ \mu=0,\ 1,\ \ldots,\ \nu-1),\end{gathered} \tag{11}$$

which follow from the results presented in [2]. Now, applying to Eq. (6) the differential operator $D_{\nu\mu}$, with consideration of Eq. (11), we obtain the desired system

$$\begin{gathered}t_{\nu\mu} = D_{\nu\mu}t_0 = \sum_{p,\ q}\frac{1}{r_{pq}^{\nu+1}}X_\nu^\mu(\theta_{pq},\ \varphi_{pq})\quad (\nu=1,\ 2,\ldots;\\ \mu=0,\ \pm 1,\ \pm 2\ldots,\ \pm(\nu-1)).\end{gathered} \tag{12}$$

Negative values of the index μ are also included here, while $t_{\nu,-\mu} = (-1)^\mu \bar{t}_{\nu\mu}$, since $p_\nu^{-\mu}(u) = (-1)^\mu p_\nu^\mu(u)$, and the bar above a quantity denotes its complex conjugate. We find the representation of the functions $t_{\nu\mu}$ by a double Fourier series from Eq. (7) with the aid of the operator $D_{\nu\mu}$:

$$\begin{gathered}t_{\nu\mu} = \begin{cases}\sum\limits_{m,\ n}\xi_{mn}^{\nu\mu}\exp[-\delta_{mn}z + i(\alpha_m x+\beta_n y)],\ z>0,\\ (-1)^{\nu+\mu}\sum\limits_{m,\ n}\xi_{mn}^{\nu\mu}\exp[\delta_{mn}z + i(\alpha_m x+\beta_n y)],\ z<0,\end{cases}\\ \xi_{mn}^{\nu\mu} = \sqrt{\frac{2(2\nu+1)}{(\nu+\mu)!\ (\nu-\mu)!}}\ \frac{\pi}{ab}\ \delta_{mn}^{\nu-\mu-1}(\beta_n - i\alpha_m)^\mu,\\ \xi_{00}^{10} = \frac{\sqrt{6}\pi}{ab},\quad \xi_{00}^{\nu\mu} = 0\,(\nu\neq 1,\ \mu\neq 0)\quad (\nu=1,2,\ldots;\\ \mu=0,\pm 1,\ldots,\pm(\nu-1)).\end{gathered} \tag{13}$$

From the complex functions $t_{\nu\mu}$ we may generate real expressions, even in x and y, viz.: $t_{\nu,2k} + \bar{t}_{\nu,2k} = t_{\nu,2k} + t_{\nu,-2k}$, which will be used below.

Equation (13), the solutions $t_{\nu\mu}$ of the Laplace equation, can satisfy boundary conditions (4) on the plane edges of the grainy layer. For fulfillment of the thermal contact conditions between layer and grains, Eq. (2), $t_{\nu\mu}$ must be transformed to a local spherical coordinate system. This can be done commencing from Eq. (12) and the addition theorem for external solutions of the Laplace equation in spherical coordinates [3]

$$\frac{1}{r_{pq}^{\nu+1}}X_\nu^\mu(\theta_{pq},\ \varphi_{pq}) = \sum_{t=0}^\infty\sum_{s=-t}^t \beta_{\nu t}^{\mu s}\gamma_{\nu t}^{\mu s}(p,\ q)\,r^t X_t^s(\theta,\ \varphi)\quad (r<a,\ b), \tag{14}$$

$$\beta_{\nu t}^{\mu s} = (-1)^{t+s}\left[\frac{2(2\nu+1)(\nu+t+\mu-s)!\,(\nu+t-\mu+s)!}{(2t+1)(2\nu+2t+1)(\nu+\mu)!\,(\nu-\mu)!\,(t+s)!\,(t-s)!}\right]^{1/2} p_{t+\nu}^{\mu-s}(0),$$

$$\gamma_{\nu t}^{\mu s}(p, q) = \frac{\exp[i(\mu-s)\varphi_{pq}^{0}]}{(p^2a^2+q^2b^2)^{\frac{1}{2}(\nu+t+1)}}, \quad \exp(i\varphi_{pq}^{0}) = -\frac{pa+iqb}{(p^2a^2+q^2b^2)^{1/2}}.$$

We then arrive at the following equation:

$$t_{\nu\mu}(r, \theta, \varphi) = \frac{1}{r^{\nu+1}} X_{\nu}^{\mu}(\theta, \varphi) + \sum_{t=0}^{\infty}\sum_{s=-t}^{t} \eta_{\nu t}^{\mu s} r^t X_t^s(\theta, \varphi) \quad (r < a, b),$$

$$\eta_{\nu t}^{\mu s} = [1+(-1)^{\mu-s}]\beta_{\nu t}^{\mu s}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \epsilon_{pq}[\gamma_{\nu t}^{\mu s}(p, q) + \gamma_{\nu t}^{\mu s}(p, -q)], \tag{15}$$

$$\epsilon_{00} = 0, \; \epsilon_{0q} = \frac{1}{2}(q>0), \; \epsilon_{p0} = \frac{1}{2}(p>0), \; \epsilon_{pq} = 1 \; (p, q > 0).$$

Since $[1 + (-1)^{\mu-s}]\beta_{\nu t}^{\mu s} \neq 0$ only for $\mu - s$ and $t + \nu$ even, in the first equation of Eq. (15) both ν and t as well as μ and s have identical parity.

3. As follows from the formulation of the problem, the temperature field in the given grainy layer is an even function of the variables x and y, and the temperature distribution within all grains is identical, i.e., in Eq. (5) the indices pq of the undetermined constants $D_{\nu\mu}^{(pq)}$ may be omitted. In this case $D_{\nu,2k+1} = 0$, $D_{\nu,2k} = D_{\nu,-2k}$. We now write solution (3) in the chosen layer in the form

$$T = \gamma z + \sum_{\nu=1}^{\infty}\sum_{\mu=-\nu+1}^{\nu-1} A_{2\nu-1,\,2\mu} t_{2\nu-1,\,2\mu} + \sum_{m,n}{}' \{B_{mn}\exp[(z-h_1)\delta_{mn}] + C_{mn}\exp[-(z+h_2)\delta_{mn}]\}\exp[i(\alpha_m x + \beta_n y)], \tag{16}$$

where the prime superscript on a summation denotes that it omits the term with indices m = n = 0.

To satisfy condition (4) we transform Eq. (16) to the coordinates x, y, z, using the expansions of Eq. (13). As a result, we obtain algebraic expressions which we solve for B_{mn} and C_{mn}, obtaining

$$\gamma c + \frac{2\sqrt{6}\pi}{ab} A_{1,0} = d,$$

$$B_{mn} = \frac{-\exp(-h_2\delta_{mn})}{1-\exp(-c\delta_{mn})}\sum_{\nu=1}^{\infty}\sum_{\mu=-\nu+1}^{\nu-1} A_{2\nu-1,\,2\mu}\,\xi_{mn}^{2\nu-1,\,2\mu}, \tag{17}$$

$$C_{mn} = \frac{\exp(-h_1\delta_{mn})}{1-\exp(-c\delta_{mn})}\sum_{\nu=1}^{\infty}\sum_{\mu=-\nu+1}^{\nu-1} A_{2\nu-1,\,2\mu}\,\xi_{mn}^{2\nu-1,\,2\mu}.$$

With consideration of Eq. (17), Eq. (16) takes on the form

$$T = \gamma z + \sum_{\nu=1}^{\infty}\sum_{\mu=-\nu+1}^{\nu-1} A_{2\nu-1,\,2\mu} t_{2\nu-1,\,2\mu} - 2\sum_{m,n}{}' \xi_{mn}\,\mathrm{sh}(z\delta_{mn})\exp[i(\alpha_m x +$$

$$+\beta_n y)], \; \xi_{mn} = [\exp(c\delta_{mn})-1]^{-1}\sum_{\nu=1}^{\infty}\sum_{\mu=-\nu+1}^{\nu-1} A_{2\nu-1,2\mu}\xi_{mn}^{2\nu-1,2\mu}. \tag{18}$$

This representation does not contain the quantities h_1 and h_2 and is valid in any layer, for the points of which $|z| < c$.

The right side of Eq. (18) is transformed to spherical coordinates r, θ, φ with consideration of Eq. (15) and the following equation:

$$\exp[\pm\delta_{mn}z + i(\alpha_m x + \beta_n y)] = \sum_{t=0}^{\infty}\sum_{s=-t}^{t} {}^{\pm}x_{ts}^{mn}\, r^t X_t^s(\theta, \varphi),$$

$$\begin{matrix} {}^{+}x_{ts}^{mn} \\ {}^{-}x_{ts}^{mn} \end{matrix} = \frac{\sqrt{2}\,\delta_{mn}^{t}\exp(-is\varepsilon_{mn})}{\sqrt{(2t+1)(t+s)!\,(t-s)!}}\begin{cases} i^s, \\ (-1)^t i^{-s}, \end{cases}$$

which is obtained after expansion of the exponential on the left in a power series and use of the corresponding integral representation of the Legendre joining functions. We then obtain

$$T(r,\theta,\varphi)=\sqrt{\frac{2}{3}}\,\gamma r X_1^0(\theta,\varphi)+\sum_{t=1}^{\infty}\sum_{s=-t+1}^{t-1}\left[A_{2t-1,2s}\frac{1}{r^{2t}}+a_{2t-1,2s}\,r^{2t-1}\right]X_{2t-1}^{2s}(\theta,\varphi), \tag{19}$$

$$a_{2t-1,2s}=\sum_{\nu=1}^{\infty}\sum_{\mu=-\nu+1}^{\nu-1}A_{2\nu-1,2\mu}\eta_{2\nu-1,2t-1}^{2\mu,2s}-2\sum_{m,n}{}'+x_{2t-1,2s}^{mn}\xi_{mn}\ (r<a,b).$$

Satisfaction of the contact conditions (2) by Eqs. (19) and (5) leads to an infinite system of algebraic equations, which we write in the form

$$\frac{1}{R^{2t}}A_{2t-1,2s}+R^{2t-1}a_{2t-1,2s}+\sqrt{\frac{2}{3}}\,\gamma R\delta_{2t-1}^{1}\delta_{2s}^{0}=R^{2t-1}D_{2t-1,2s}, \tag{20}$$

$$\left[2t\left(1+\frac{\lambda_1}{\lambda_2}\right)-1\right]\frac{1}{R^{2t+1}}A_{2t-1,2s}+\left(1-\frac{\lambda_1}{\lambda_2}\right)(2t-$$

$$-1)R^{2t-2}\sum_{\nu=1}^{\infty}\sum_{\mu=-\nu+1}^{\nu-1}A_{2\nu-1,2\mu}\left[\eta_{2\nu-1,2t-1}^{2\mu,2s}-y_{2\nu-1,2t-1}^{2\mu,2s}\right]=\sqrt{\frac{2}{3}}\,\gamma\delta_{2t-1}^{1}\delta_{2s}^{0}\left(1+\frac{\lambda_1}{\lambda_2}-2t\right); \tag{21}$$

$$y_{2\nu-1,2t-1}^{2\mu,2s}=2\sum_{m,n}{}'+x_{2t-1,2s}^{mn}\left[\exp(c\delta_{mn})-1\right]^{-1}\xi_{mn}^{2\nu-1,2\mu}.$$

The second group of equations (21) is a closed infinite system with unknowns $A_{2t-1,2s}$ ($t = 1, 2, \ldots; s = 0, 1, \ldots, t-1$), where $A_{2t-1,-2s} = A_{2t-1,2s}$. The first group of equations in Eq. (20) is an expression of the unknowns $D_{2t-1,2s}$ in terms of $A_{2\nu-1,2\mu}$.

In this manner, the undetermined constants in Eq. (18) must be found from the infinite algebraic system. Its properties are determined by the behavior of the quantities $|\eta_{2\nu-1,2t-1}^{2\mu,2s}|$ and $|y_{2\nu-1,2t-1}^{2\mu,2s}|$ as $\nu, t\to\infty$ for $0\leqslant\mu\leqslant\nu-1$, $0\leqslant s\leqslant t-1$. We have the following upper limits for these quantities:

$$|\eta_{2\nu-1,\ 2t-1}^{2\mu,2s}|<18\sqrt{\frac{4\nu-1}{4t-1}}\,\frac{(2\nu+2t-2)!}{(2\nu-1)!\,(2t-1)!}\left[\frac{\sqrt{a}}{a^{\nu+t}}+\frac{\sqrt{b}}{b^{\nu+1}}\right]^2,$$

$$|y_{2\nu-1,\ 2t-1}^{2\mu,2s}|<K\frac{(2\nu+2t-2)!}{(2\nu-1)!\,(2t-1)!}\sqrt{\frac{4\nu-1}{4t-1}}\,\frac{1}{c^{2t+2\nu}},$$

where K is some constant independent of ν, t, μ and s. Using these estimates and transforming in Eq. (21) to new unknowns $z_{2t-1,2s} = (1/R^{2t+1})A_{2-1,2s}$, it is simple to show that the system obtained after the replacement belongs to the class of normal type systems, if a, b, c > 2R. Since satisfaction of these inequalities was presumed in the formulation of the problem (the grains do not contact each other), the possibility of obtaining a solution of system (21) by the reduction method has been proven.

4. The results presented above are sufficient for determination of effective thermal conductivity coefficients of the grainy layer. The mean projections (over elementary cell volume of the medium) on the z axis of the temperature gradient $\partial T/\partial z$ and the thermal flux vector q_z can be represented by inequalities

$$abc\left\langle\frac{\partial T}{\partial z}\right\rangle=\int_{V_M}\frac{\partial T}{\partial z}dV+\int_{V_g}\frac{\partial T_{00}}{\partial z}dV,$$
$$-abc\,\langle q_z\rangle=\lambda_1\int_{V_M}\frac{\partial T}{\partial z}dV+\lambda_2\int_{V_g}\frac{\partial T_{00}}{\partial z}dV, \tag{22}$$

where $V_M + V_g = abc$; $V_g = (4/3)\pi R^3$.

Using the Gauss–Ostrogradskii theorem, the temperature field equations (5) and (19), the first equations of Eqs. (4) and (17), the equations of the infinite system Eqs. (20), (21) at t = 1, s = 0, and the property of orthogonality of the spherical harmonics $X_\nu^\mu(\theta, \varphi)$, we transform Eq. (22) to

$$abc\left\langle \frac{\partial T}{\partial z}\right\rangle = \gamma abc + 2\sqrt{6}\pi A_{1,0}, \quad -abc\,\langle q_z\rangle = \lambda_1\gamma abc. \tag{23}$$

Having defined the effective thermal conductivity coefficient λ_{ef} as the ratio of $-\langle q_z\rangle$ to $\left\langle \frac{\partial T}{\partial z}\right\rangle$, we obtain from Eq. (23)

$$\lambda_{ef} = \lambda_1 \frac{1}{1 + \frac{2\sqrt{6}\pi}{abc} A_{1,0}^{(1)}}, \tag{24}$$

where $A_{1,0}^{(1)}$ denotes the first unknown $A_{1,0}$ of infinite system (21) at $\gamma = 1$.

Thus, to find the effective thermal conductivity λ_{ef} it is necessary to have the solution of system (21). This solution, as was shown in Sec. 3, can be found to the required accuracy from a truncated system, the order of which depends on the values of the parameters a/R, b/R and c/R. If as a zeroth approximation we confine ourselves to one equation in system (21), then

$$A_{1,0}^{(1)} = \sqrt{\frac{2}{3}}\, R^3 \frac{\frac{\lambda_1}{\lambda_2} - 1}{1 + 2\frac{\lambda_1}{\lambda_2} + \left(1 - \frac{\lambda_1}{\lambda_2}\right) R^3 \left(\eta_{11}^{00} - y_{11}^{00}\right)}. \tag{25}$$

Considering the notation used in Eq. (15) and system (21), we write Eq. (24), with consideration of Eq. (25), in its final form

$$\lambda_{ef} = \lambda_1 \frac{1}{1 + 3f\omega_1},$$

$$\omega_1 = \frac{\frac{\lambda_1}{\lambda_2} - 1}{1 + 2\frac{\lambda_1}{\lambda_2} + 3\left(\frac{\lambda_1}{\lambda_2} - 1\right) f\omega_2}, \quad f = \frac{4\pi R^3}{3abc}, \tag{26}$$

$$\omega_2 = \sum_{m,n} \frac{c\delta_{mn}}{\exp(c\delta_{mn}) - 1} - \frac{1}{\pi}\sum_{p,q=0}^{\infty} \epsilon_{pq} \frac{abc}{(p^2a^2 + q^2b^2)^{3/2}}.$$

As follows from these equations, the ratio of λ_{ef} to λ_1 depends not only on the grain concentration f and the ratio of the thermal conductivities of matrix λ_1 and grains λ_2, but also on the parameters a/c and b/c, since $\omega_2 = \omega_2(a/c, b/c)$. The values of the quantity ω_2 are presented in Table 1.*

The first approximation of Eq. (26) can give satisfactory results only at small values of the quantities $|\lambda_1/\lambda_2 - 1|$ and f. In fact, at $\lambda_1/\lambda_2 - 1 = 0$ the infinite system of Eq. (21) degenerates into a single equation $A^{(1)} = 0$, and we obtain from Eq. (26) the trivial result $\lambda_{ef} = \lambda_1$ for arbitrary f. But if $\lambda_1/\lambda_2 - 1 \neq 0$, then the unknown $A_{1,0}^{(1)}$ in Eq. (25) can differ slightly from its exact value only at $f \ll \pi/6$, i.e., when the mutual interaction

*The calculations were performed by A. G. Artemenko, to whom the author expresses his gratitude.

TABLE 1. Values of the Function $\omega_2 = \omega_2(a/c, b/c)$

b/c	a/c 0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
0,6	—1,18	—1,06	—1,02	—1,00	—1,00	—1,00	—1,00	—1,00
0,8	—1,06	—0,87	—0,78	—0,72	—0,67	—0,62	—0,57	—0,53
1,0	—1,02	—0,78	—0,66	—0,57	—0,49	—0,42	—0,34	—0,27
1,2	—1,00	—0,72	—0,57	—0,46	—0,36	—0,27	—0,18	—0,09
1,4	—1,00	—0,67	—0,49	—0,36	—0,25	—0,14	—0,03	0,07
1,6	—1,00	—0,62	—0,42	—0,27	—0,14	—0,01	0,11	0,23
1,8	—1,00	—0,57	—0,34	—0,18	—0,03	0,11	0,25	0,38
2,0	—1,00	—0,53	—0,27	—0,09	0,07	0,23	0,38	0,54

of the grain temperature fields is insignificant. To establish the dependence of the accuracy of Eq. (25) on the values of the parameters $\lambda_1/\lambda_2 - 1$ and f is difficult, so we will present a comparison of results calculated with Eq. (26) with data of other authors:

Reference	[4]	[5]	[6]	[6]	[7]	[26]	[4]
λ_{ef}/λ_1	2,94	2,15	4,63	2,53	2,88	2,90	3,32

This comparison is for quartz type grainy material in water with parameters $\lambda_1 = 0.63$ W/m·deg, $\lambda_2 = 8.36$ W/m·deg, and f = 0.48 [4], with the bottom row showing experimental values from [4]. Moreover, the first and second approximations from [6] and the first Rayleigh approximation from [7] were used. In the calculations with Eq. (26) it was assumed that $a/c = b/c = 1$, i.e., $\omega_2 = -0.66$. Thus, the first approximation of the theory proposed gives completely satisfactory results, even for cases of significant difference between thermal conductivities of matrix and grain materials and relatively high grain concentrations f.

To obtain a more exact value of λ_{ef} it is necessary to substitute in Eq. (24) a more exact value of the unknown $A_{1,0}^{(1)}$, i.e., to retain a larger number of equations in system (21). This would correspond to a stricter satisfaction of contact conditions (2). We note, however, that even the first approximation captures such a significant property of the thermal conductivity of grainy materials as its anisotropy. In fact, if in the results presented above we perform a cyclical interchange of the coordinates x, y, z and parameters a, b, c, we arrive at the formulas for the thermal conductivity coefficients in the x and y directions. Since ω_2 changes quite significantly, in the general case $\lambda_{ef}^{(z)} \neq \lambda_{ef}^{(y)} \neq \lambda_{ef}^{(x)}$.

In conclusion, we note that in the present study the thermal conductivity coefficient of the grainy medium was determined by strict solution of the boundary problem of Eqs. (1), (2). Also of great importance is the proof of the possibility of finding the unknown $A_{1,0}^{(1)}$ from system (21) to the required degree of accuracy. The proposed method is in all probability the simplest possible one using a strict solution of the corresponding boundary problem, since it employs series. Only double series were used, while in Rayleigh's approximate approach and that of other authors refining his results, triple series were used [7]. Another approach to determination of effective thermal conductivity was proposed in [7], in which the problem is reduced to solution of an infinite system of algebraic equations. We also note that the results presented in the first part of this study can be used for solution of certain thermal conductivity problems for a grainy layer or a rectangular parallelepiped containing a spherical inclusion. Moreover, this method permits generalization to the case of a grainy material with arbitrary nonorthogonal inclusion lattice.

NOTATION

λ, thermal conductivity; R, grain radius; a, b, c, lattice periods; x, y, z, Cartesian coordinates; r, θ, φ, spherical coordinates; T, temperature.

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